D. Pivoting

1. Since we wish to move from one basic feasible solution to an adjacent bfs, we do so by moving from one extreme point to an adjacent extreme point. This corresponds to interchanging one basic variable x_p with one non-basic variable x_q. x_p is called the **LEAVING VARIABLE** while x_q is called the **ENTERING VARIABLE**.

e.g. $\underline{a}^1 \; \underline{a}^2 \; \ldots \underline{a}^m \; \underline{a}^{m+1} \; \ldots \underline{a}^n$

Suppose $\underline{a}^1 \; \underline{a}^2 \; \ldots \underline{a}^m$ is a basis having a bfs. It is possible that say, $\underline{a}^{m+3}$, will enter the basis and $\underline{a}^2$ leaves. (Of course this is only possible if $\underline{a}^1 \; \underline{a}^3 \; \ldots \underline{a}^m \; \underline{a}^{m+3}$ is a basis and the solution corresponding to this solution is basic feasible.)

Question: How do we choose the entering variable?

Solution: It must improve the value of the objective function when its value is increased from 0.

Note: The value of the leaving variable becomes 0.

Example 4.17:

$$\begin{array}{cccccc} \underline{a}^1 & \underline{a}^2 & \underline{a}^3 & \underline{a}^4 & \underline{a}^5 & \underline{b} \\ 1 & 0 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & 4 & -3 & 2 \\ 0 & 0 & 1 & 3 & 4 & 6 \end{array}$$

Basic feasible solution: $(1,2,6,0,0)$

Suppose we want to being $\underline{a}^5$ into the basis. (Note that $\{\underline{a}^1, \underline{a}^2, \underline{a}^3, \underline{a}^5\}$ is a linearly dependent set but possibly we can find a lin. indep. subset containing $\underline{a}^5$.)

Step 1: Compute the ratios b_i / a_{i5} .

i.e. $1/1, 2/-3, 6/4$.

Step 2: Select the smallest nonnegative ratio above.

e.g. 1 corresponding to b_1 / a_{15}

so a_{15} is the pivot element.

Note: Choosing the smallest nonegative one as pivot ensures that the new solution will be feasible as well as basic.

Step 3: Pivot (Gauss-Jordan) around a_{15}.

a^1	a^2	a^3	a^4	a^5	b
1	0	0	−2	1	1
3	1	0	−2	0	5
−4	0	1	11	0	2

$\underline{a}^2, \underline{a}^3, \underline{a}^5$ is our new basis yielding a bfs $(0,5,2,0,1)$.

It is adjacent to the bfs $\left(1,2,6,0,0\right)$ since the basic variables differ only by one entering and one leaving variable.

Note that all of the columns of the basic variable are 0 except for a single 1; i.e. we are using the Euclidean coordinate basis. This is because the tableau was in canonical form.

Def. 4.17: A tableau is said to be in CANONICAL FORM iff among the n variables are m basic variables such that

(1) for each basic variable, there is exactly one equation in the tableau wherein the coefficient of the basic variable is nonzero;

(2) the coefficients in the tableau of the basic variables are

either 0 or 1; and

(3) no two basic variables in the same row (equation) can have coefficient 1.

E. Primal Simplex Algorithm

1. assumptions:

a. The problem is in standard form

b. A bfs is known (initial or starting bfs)

c. The tableau is in canonical form w.r.t the initial b.f.s.

Suppose $\left(\mathbf{x_B}, \underline{\mathbf{0}}\right)$ is a basic feasible solution.

Canonical form:

$$
\begin{array}{cccccccc}
\underline{a}^{1} & \underline{a}^{2} & \cdots & \underline{a}^{m} & \underline{a}^{m+1} & \cdots & \underline{a}^{n} & \underline{b} \\
1 & 0 & & 0 & a_{1m+1} & \cdots & a_{1n} & b_{1} \\
0 & 1 & & 0 & a_{2m+1} & \cdots & a_{2n} & b_{2} \\
0 & 0 & & 0 & \vdots & & \vdots & \vdots \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & & 1 & a_{mm+1} & & a_{mn} & b_{m}
\end{array}
$$

Obj. function: $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Temporarily ignore the bfs $\left(x_B, \underline{0}\right)$ **.**

Instead, solve for the basic variables $x_1, x_2, \ldots, x_m$ **in terms of the non-basic variables** $x_{m+1}, \ldots, x_n$ **.**

$$
\begin{aligned}
x_1 &= b_1 - \sum_{j=m+1}^{n} a_{1j} x_j \\
x_2 &= b_2 - \sum_{j=m+1}^{n} a_{2j} x_j \\
\vdots \quad & \vdots \quad \vdots \quad \vdots \qquad \vdots \\
x_m &= b_m - \sum_{j=m+1}^{n} a_{mj} x_j
\end{aligned}
$$

or in matrix form

$$\underline{x}_B = \underline{b} - D\underline{x}_D$$ where

$$
D = \begin{pmatrix} a_{1m+1} & \cdots & a_{1n} \\ a_{2m+1} & \cdots & a_{2n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{mm+1} & & a_{mn} \end{pmatrix}
$$

Hence, for any feasible solution,

$$z = \underline{c} \cdot \mathbf{x} = c_1(b_1 - \sum_{j=m+1}^{n} a_{1j}x_j) + \ldots c_m(b_m - \sum_{j=m+1}^{n} a_{mj}x_j)$$

$$+ c_{m+1}x_{m+1} + \ldots + c_n x_n$$

$$= c_1b_1 + c_2b_2 + \ldots + c_m b_m - \sum_{k=1}^{m} c_k \left(\sum_{j=m+1}^{n} a_{kj}x_j \right) + c_{m+1}x_{m+1} + \ldots + c_n x_n$$

From the canonical tableau, at the bfs $\mathbf{x}_0 \equiv (\mathbf{x}_B, \underline{0})$, we have

$$x_1 = b_1, \ldots, x_m = b_m, x_{m+1} = x_{m+2} = \ldots = x_n = 0$$

so that at this bfs, the value of the objective function, which we denote by z_0 is given by

$$z_0 = c_1b_1 + c_2b_2 + \ldots + c_m b_m + (c_{m+1} \cdot 0 + \ldots + c_n \cdot 0)$$

$$= c_1b_1 + c_2b_2 + \ldots + c_m b_m$$

In other words, for any feasible solution, the value of the objective function can be written in terms of the value of the objective function at the starting bfs $\mathbf{x}_0$ as

$$z = z_0 + c_{m+1}x_{m+1} + \ldots + c_n x_n - \sum_{j=m+1}^{n} x_j \left(\sum_{k=1}^{m} c_k a_{kj} \right)$$

$$= z_0 + \left(c_{m+1} - z_{m+1}\right)x_{m+1} + \left(c_{m+2} - z_{m+2}\right)x_{m+2} + \ldots + \left(c_n - z_n\right)x_n$$

where

$$\boxed{z_j \equiv \sum_{k=1}^{m} a_{kj} c_k}$$

In matrix form

$$\boxed{z_j = \underline{c}_B \cdot \underline{a}^j}$$

Def. 4.18: $r_j = c_j - z_j$ is called the **RELATIVE COST COEFFICIENT**, or equivalently, the **REDUCED COST COEFFICIENT**. If x_j is a basic variable , then $r_j = 0$. If x_j is a non-basic variable, then the reduced cost r_j is the amount the objective function would be changed if the variable x_j were increased by 1 (one unit) while holding all the other non-basic variables at 0. We may also note the interpretation: (revenue per unit of item j)–(cost of resources per unit of item j)

Def. 4.19: $\boxed{z_0 = \sum_{k=1}^{m} b_k c_k = c_B \cdot x_B}$ is our notation for the objective function evaluated at the initial bfs.

Def. 4.20: Adjoining a row of the reduced costs and $-z_0$ to the canonical form is then the **CANONICAL SIMPLEX TABLEAU**.

So suppose $(\underline{x}_B, \underline{0})$ is a basic feasible solution (bfs).

<u>Canonical simplex tableau</u>:

$$
\begin{array}{cccccccc}
\underline{a}^1 & \underline{a}^2 & \cdots & \underline{a}^m & \underline{a}^{m+1} & \cdots & \underline{a}^n & \underline{b} \\
1 & 0 & & 0 & a_{1m+1} & \cdots & a_{1n} & b_1 \\
0 & 1 & & 0 & a_{2m+1} & \cdots & a_{2n} & b_2 \\
0 & 0 & & 0 & \vdots & & \vdots & \vdots \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & & 1 & a_{mm+1} & & a_{mn} & b_m \\
\hline
0 & 0 & \cdots & 0 & r_{m+1} & \cdots & r_n & -z_0
\end{array}
$$

under the assumption that the basic variables are $x_1, \ldots, x_m$ and the non-basic variables are $x_{m+1}, \ldots, x_n$.

SIMPLEX ALGORITHM

Step 1: Calculate $z_j, r_j, \text{and } z_0$. We do this indirectly as we put the tableau into canonical form by Gauss-Jordan reduction with the first line beginning with the original costs. We do have to calculate z_0 .

Step 2: Test the reduced cost coefficients r_j .

(i) If $r_j \geq 0$ for all j=1,...,n then STOP – the current bfs is optimal.

(ii) If not, let $J = \left\{ j : r_j < 0 \right\}$. We will examine this set J further in the next step.

Step 3: Test the $\underline{a}^j$ for $j \in J$.

(i) If $\underline{a}^j \leq \underline{0}$, $\underline{a}^j \neq \underline{0}$ for any $j \in J$ then STOP – the objective function is UNBOUNDED BELOW on the constraint set and no optimal feasible solution exists.

(ii) Otherwise, find the index $j_0 \in J$ such that the magnitude of the reduced cost r_{j_0} is greater than or equal to the magnitude of any other reduced cost r_j, $j \in J$; i.e.

$$\boxed{\left| r_{j_0} \right| = \max_{j \in J} \left| r_j \right| = \max_{j \in J} \left| c_j - z_j \right|}$$

This is the entry criterion which determines which vector will enter the basis; i.e. $\underline{a}^{j_0}$ will enter the basis. Equivalently, the non-basic variable x_{j_0} will become basic.

(iii) Determine the index l_0 such that $\boxed{\dfrac{b_{l_0}}{a_{l_0 j_0}} = \min_{\substack{i \\ \text{s.t.} \\ a_{i j_0} > 0}} \left\{ \dfrac{b_i}{a_{i j_0}} \right\}}$

i.e. we select the smallest non-negative ratio. This is called the **EXIT CRITERION**. Then $\underline{a}^{l_0}$ leaves the basis.

(Note: This is where we have the first m columns basic. We should therefore put " " around l_0 because we will have to reorder the columns to keep the first m columns correspond to the basic and the remaining to the non-basic. We will see this better later in an example.

<u>Step 4:</u> Pivot on $a_{l_0 j_0}$. This yields a new b.f.s.

<u>Step 5:</u> Repeat the algorithm from Step 2.

Note: The " " means refer to the reordered columns so that the 1st m columns form an m x m identity matrix.

Example 4.18 (Reddy Mikks Co.)

$$\begin{array}{lllll} \text{Maximize} & z & = & 3x_E & + \; 2x_I \\ \text{s.t.} \end{array}$$

$$\begin{array}{rcrcrl} x_E & + & 2x_I & \leq & 6 & \quad (1) \\ 2x_E & + & x_I & \leq & 8 & \quad (2) \\ -x_E & + & x_I & \leq & 1 & \quad (3) \\ & & x_I & \leq & 2 & \quad (4) \end{array}$$

$$x_I \geq 0, \quad x_E \geq 0$$

In standard form,

$$\text{Minimize} \quad z = -3x_E - 2x_I$$

s.t.

$$\begin{aligned}
x_E + 2x_I + y_1 &= 6 \quad (1)\\
2x_E + x_I + y_2 &= 8 \quad (2)\\
-x_E + x_I + y_3 &= 1 \quad (3)\\
x_I + y_4 &= 2 \quad (4)
\end{aligned}$$

$$x_I \geq 0, \quad x_E \geq 0 \quad y_i \geq 0 \quad i = 1,\ldots,4$$

$\underline{a}^1$	$\underline{a}^2$	$\underline{a}^3$	$\underline{a}^4$	$\underline{a}^5$	$\underline{a}^6$	$\underline{b}$
1	2	1	0	0	0	6
2	1	0	1	0	0	8
–1	1	0	0	1	0	1
0	1	0	0	0	1	2

$\underline{a}^3, \ \underline{a}^4, \ \underline{a}^5, \ \underline{a}^6$ form a basis so our initial basic variables are y_1, y_2, y_3, y_4 and the non-basic variables are x_I, x_E .

initial b.f.s.: $\left(0,\ 0,\ 6,\ 8,\ 1,\ 2\right)$ We consider the constraint in the form $c_1x_1 + c_2x_2 + \ldots + c_nx_n - z = 0$ and consider a new variable –z which becomes one of our initial basic variables. $z_0 = -3 \times 0 + -2 \times 0 = 0$

Tableau 0:

Basic	x_E	x_I	y_1	y_2	y_3	y_4	$-z$	Solution
$-z$	-3	-2	0	0	0	0	1	0
y_1	1	2	1	0	0	0	0	6
y_2	2	1	0	1	0	0	0	8
y_3	-1	1	0	0	1	0	0	1
y_4	0	1	0	0	0	1	0	2

Remark: We have shown that $z = z_0 + \sum_{j=m+1}^{n} r_j x_j$ where

we have ordered the columns so that $\underline{a}^1 \ldots \underline{a}^m$ are basic and $\underline{a}^{m+1} \ldots \underline{a}^n$ are non-basic. This formula shows the reduced costs for the basic variables are 0.

Reduced cost for x_E, x_I : Note that the cost coefficients for $y_1, \ldots, y_m$ are 0 so that z_E, z_I are 0. Hence

$$r_E = -3 - 0 = -3$$

and

$$r_I = -2 - 0 = -2$$

Since Tableau 0 is in canonical form (except for reordering of the columns) we know that the numbers in the constraint row are the reduced costs.

We note that not all of the reduced costs are 0 so we will have to find the entering and leaving variables. From the Simplex Algorithm, we take x_E as the entering variable since it has the largest absolute value of the negative reduced costs.

Note 2: We calculate the ratios for which the elements of the x_E column are greater than 0; i.e. $\frac{6}{1} = 6; \frac{8}{2} = 4$ so that we pivot on 2 – the leaving variable is y_2 since the canonical tableau can only have one column with only a 1 in the 3rd row.

<u>Tableau 1:</u>

Basic	x_E	x_I	y_1	y_2	y_3	y_4	$-z$	Solution
$-z$	0	$-\frac{1}{2}$	0	$\frac{3}{2}$	0	0	1	+12
y_1	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	0	0	2
x_E	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	4
y_3	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	0	0	5
y_4	0	1	0	0	0	1	0	2

Note that x_I is the only variable with a negaive reduced cost so it will enter the basis. Calculate the ratios for the elements in this column which are greater than 0:

$\frac{2}{3/2} = \frac{4}{3}$; $\frac{4}{1/2} = 8$; $\frac{5}{3/2} = \frac{10}{3}$; **The smallest ratio is the first one so y_1 is leaving the basis. Pivot on $\frac{3}{2}$.**

Tableau 2:

Basic	x_E	x_I	y_1	y_2	y_3	y_4	$-z$	Solution
$-z$	0	0	$\frac{1}{3}$	$\frac{4}{3}$	0	0	1	$+12\,{}^{2}\!/_{3}$
x_I	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	$\frac{4}{3}$
x_E	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	$\frac{10}{3}$
y_3	0	0	-1	1	1	0	0	3
y_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	0	$\frac{2}{3}$

STOP – All of the reduced costs are greater than or equal to 0.

From the tableau, we can read off the optimal solution and the value of the objective function at the optimal solution:

$$x_I = \frac{4}{3} \quad x_E = \frac{10}{3} \quad z = -12\,{}^{2}\!/_{3}$$

Therefore the solution to the original Reddy Mikks Co. problem

is to produce $\frac{4}{3}$ tons of interior paint and $\frac{10}{3}$ tons of exterior paint daily which will yield an optimal sales revenue of $12\frac{2}{3}$ thousand dollars per day.

Example 4.19:

$$
\begin{array}{llrrrrrrrl}
\text{Minimize} & z = & 2x_1 & - & x_2 & + & 3x_3 & - & 2x_4 & \\
\text{st} & & & & & & & & & \\
& 3x_1 & + & & x_3 & - & 5x_4 & = & 9 & (1) \\
& -x_1 & + & x_2 & & + & x_4 & = & 2 & (2) \\
& x_i \ge 0 \quad i = 1,\dots 4 & & & & & & & &
\end{array}
$$

Solution: We note that an initial bfs is given by

$$x_1 = 0, \quad x_2 = 2, \quad x_3 = 9, \quad x_4 = 0$$

x_2, x_3 are basic variables while x_1, x_4 are the non-basic variables.

Tableau 0:

Basic	x_1	x_2	x_3	x_4	Solution
x_3	3	0	1	-5	9
x_2	-1	1	0	1	2
$-z$	-8	0	0	14	-25

Note 1: We have suppressed the $-z$ column since it does not change.

Note 2: We calculated the reduced costs as follows:

$z_1 = 3c_3 + (-1)c_2 = 3 \times 3 + (-1) \times (-1) = 9 + 1 = 10$

Hence $r_1 = c_1 - z_1 = 2 - 10 = -8$.

Since x_2, x_3 **are basic,** $r_2 = r_3 = 0$.

$z_4 = (-5)c_3 + (1)c_2 = -5 \times 3 + (1) \times (-1) = -15 - 1 = -16$

Hence $r_4 = c_4 - z_4 = -2 - (-16) = -2 + 16 = 14$.

$z_0 = 2 \times 0 - 2 + 3 \times 9 - 2 \times 0 = -2 + 27 = 25$ **so** $-z_0 = -25$.

This gives us the last line of the tableau 0.

However, there is another way (row reduction) as follows:

start with

Basic	x_1	x_2	x_3	x_4	Solution
x_3	3	0	1	-5	9
x_2	-1	1	0	1	2
$-z$	2	-1	3	-2	0

Row reduce to get the reduced cost coefficients for x_2, x_3 to be 0; i.e. subtract $3 \times \text{row}\, x_3$ and add $1 \times \text{row}\, x_2$ from row $-z$ to get

Basic	x_1	x_2	x_3	x_4	Solution
x_3	3	0	1	-5	9
x_2	-1	1	0	1	2
$-z$	$2-1-3\times 3$	$-1+1$	$3-3$	$-2+1-3\times(-5)$	$0+2-3\times 9$

which reduces to Tableau 0. We will show how to write the sequence of tableaus.

Basic	x_1	x_2	x_3	x_4	Solution	
$-z$	2	-1	3	-2	0	
x_3	3	0	1	-5	9	
x_2	-1	1	0	1	2	
$-z$	-8	0	0	14	-25	Tableau 0
x_3	3	0	1	-5	9	x_1 enters
x_2	-1	1	0	1	2	x_3 leaves
$-z$	0	0	$\frac{8}{3}$	$\frac{2}{3}$	-1	Tableau 1
x_1	1	0	$\frac{1}{3}$	$-\frac{5}{3}$	3	optimal
x_2	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	5	

The optimal value of the objective function is 1 which occurs at the point (3,5,0,0).

Question: How do we get our initial bfs (ibfs) in general? (We are not always so lucky to begin our problem with a canonical tableau!)

F. Artificial Variables

When an initial bfs is not obvious, one may wish to augment the tableau with artificial variable for which an initial bfs is obvious; i.e. we try to create a canonical tableau by adding additional variables to the problem. This happens naturally when only inequalities are given; i.e. when slack/surplus variables have to be inserted.

There are two distinct methods of introducing and using artificial variables:

Method (1): The Two-Phase Method. Phase 1 consists of finding a bfs for the original problem. Phase 2 uses this bfs as the ibfs to begin the simplex procedure.

Method (2): **Big M-Method (Method of penalties) This method adds artificial variables, R_i, to the problem which then form our ibfs. However, we add $M\sum R_i$ to the objective function where M is a very big positive number. This implies that in our final solution all of the artificial variables must be 0 in order that our objective function not be penalized by M x non-zero sum. We will examine this method in detail later.**

Example 4.20: **(of 2-phase method)**

$$\min \quad z = 4x_1 + x_2 + x_3$$

s.t.

$$2x_1 + x_2 + 2x_3 = 4 \quad (1)$$

$$3x_1 + 3x_2 + x_3 = 3 \quad (2)$$

$$x_i \geq 0, \quad i = 1,2,3$$

An ibfs is not obvious so we perform the 2=phase method.

Phase 1:

$$\min \quad z = x_4 + x_5$$

s.t.

$$
\begin{array}{lllllllll}
2x_1 & + & x_2 & + & 2x_3 & + & x_4 & & = 4 \quad (1) \\
3x_1 & + & 3x_2 & + & x_3 & & & + \; x_5 & = 3 \quad (2) \\
\end{array}
$$

$$x_i \geq 0, \quad i = 1, \ldots, 5$$

Basic	x_1	x_2	x_3	x_4	x_5	Solution	Tableau
$-z$	0	0	0	1	1	0	
x_4	2	1	2	1	0	4	
x_5	3	3	1	0	1	3	
$-z$	-5	-4	-3	0	0	-7	"0"
x_4	2	1	2	1	0	4	$4/2 = 2$
x_5	3	3	1	0	1	3	$3/3 = 1$
$-z$	0	1	$-4/3$	0	$5/3$	-2	"1"
x_4	0	-1	$4/3$	1	$-2/3$	2	$2/(4/3) = 3/2$
x_1	1	1	$1/3$	0	$1/3$	1	$1/(1/3) = 3$
$-z$	0	0	0	1	1	0	"2"
x_3	0	$-3/4$	1	$3/4$	$-1/2$	$3/2$	
x_1	1	$5/4$	0	$-1/4$	$1/2$	$1/2$	

End of Phase 1.

Note 1: At the end of Phase 1, all of the artificial variables should be nonbasic. If not, then there is no bfs to the original problem. We will examine this possibility more later.

Note2: The first and last –z lines should be the same in Phase 1 because of Note 1 and the form of the Phase 1 obj. function.

Phase2:

Basic	x_1	x_2	x_3	Sol'n	Tableau
$-z$	4	1	1	0	
x_3	0	$-3/4$	1	$3/2$	
x_1	1	$5/4$	0	$1/2$	
$-z$	0	$-13/4$	0	$-7/2$	"0"
x_3	0	$-3/4$	1	$3/2$	
x_1	1	$5/4$	0	$1/2$	
$-z$	$13/5$	0	0	$-11/5$	"1"
x_3	$3/5$	0	1	$9/5$	
x_2	$4/5$	1	0	$2/5$	

End of Phase 2.

Optimal Solution: $\boxed{\begin{array}{ccc} x_1 & = & 0 \\ x_2 & = & \frac{2}{5} \\ x_3 & = & \frac{9}{5} \end{array}}$ **and the value of the objective function at this optimal point is $\frac{11}{5}$. This is the minimal value fo the objective function of th original problem.**

Example 4.20a: (M-technique)

$$\min \quad z = 4x_1 + x_2 + x_3$$

s.t.

$$2x_1 + x_2 + 2x_3 = 4 \quad (1)$$

$$3x_1 + 3x_2 + x_3 = 3 \quad (2)$$

$$x_i \geq 0, \quad i = 1,2,3$$

Step 1: Rewrite the objective function to include penalty:

$$\min \quad z = 4x_1 + x_2 + x_3 + MR_1 + MR_2$$

s.t.

$$2x_1 + x_2 + 2x_3 + R_1 = 4 \quad (1)$$

$$3x_1 + 3x_2 + x_3 + R_2 = 3 \quad (2)$$

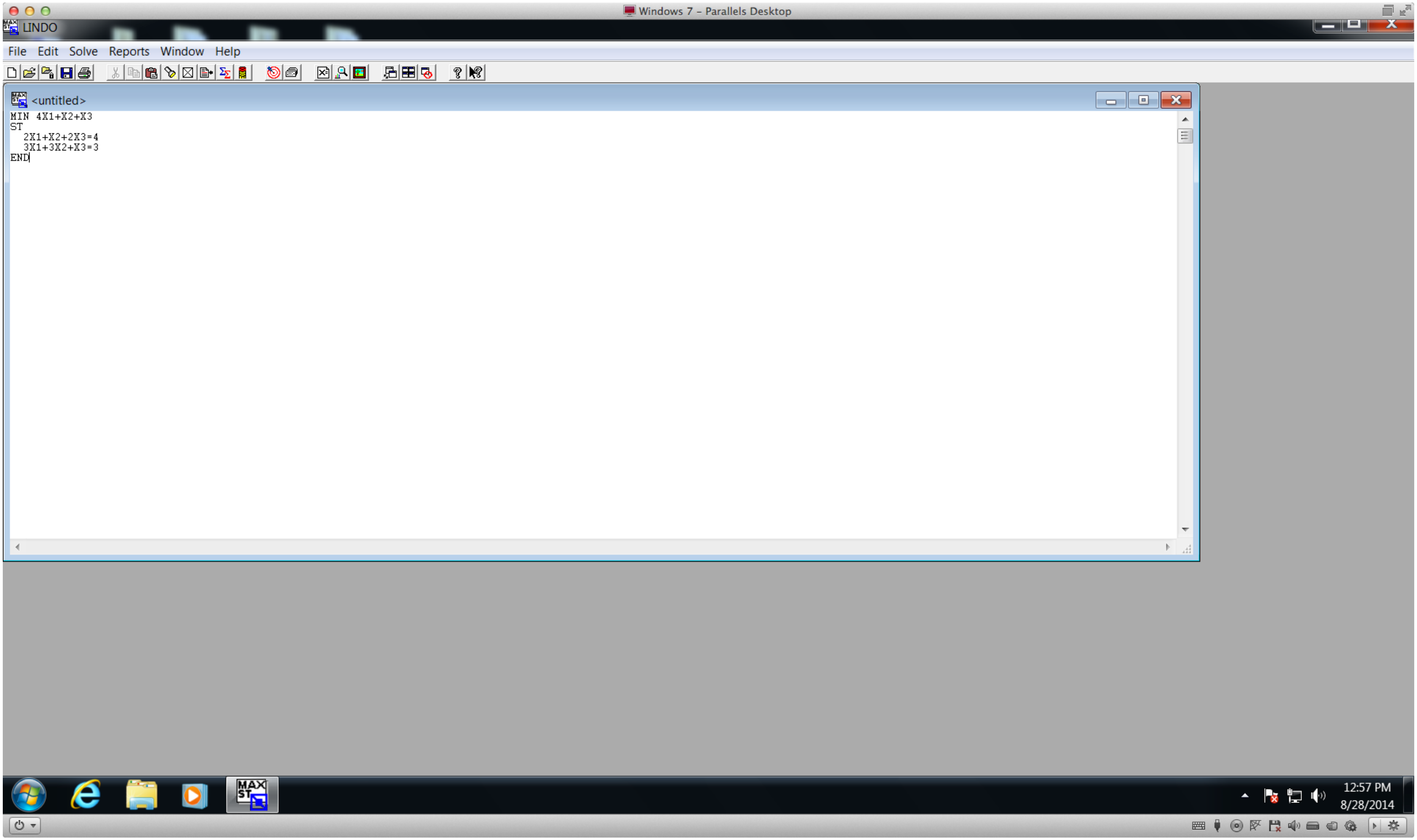
$$x_i \geq 0, \quad i = 1, 2, 3 \quad R_j \geq 0 \quad j = 1, 2$$

Basic	x_1	x_2	x_3	R_1	R_2	Sol'n.	Tableau
$-z$	4	1	1	M	M	0	
R_1	2	1	2	1	0	4	
R_2	3	3	1	0	1	3	
$-z$	$4-5M$	$1-4M$	$1-3M$	0	0	$-7M$	"0"
R_1	2	1	2	1	0	4	$\frac{4}{2}=2$
R_2	3	3	1	0	1	3	$\frac{3}{3}=1$
$-z$	0	$-3+M$	$\frac{-1-4M}{3}$	0	$\frac{-4+5M}{3}$	$-4-2M$	"1"
R_1	0	-1	$\frac{4}{3}$	1	$-\frac{2}{3}$	2	$\frac{2}{\frac{4}{3}}=\frac{3}{2}$
x_1	1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	1	$\frac{1}{\frac{1}{3}}=3$

Basic	x_1	x_2	x_3	R_1	R_2	Sol'n.	Tableau
$-z$	0	$-\frac{13}{4}$	0	$\frac{1+4M}{4}$	M	$-\frac{7}{2}$	"2"
x_3	0	$-\frac{3}{4}$	1	$\frac{3}{4}$	$\frac{-1}{2}$	$\frac{3}{2}$	
x_1	1	$\frac{5}{4}$	0	$\frac{-1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	
$-z$	$\frac{13}{5}$	0	0	$\frac{-2+5M}{5}$	$\frac{-1+5M}{5}$	$-\frac{11}{5}$	"3"
x_3	$\frac{3}{5}$	0	1	$\frac{3}{5}$	$\frac{-1}{5}$	$\frac{9}{5}$	optimal
x_2	$\frac{4}{5}$	1	0	$\frac{-1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	

Hence the optimal bfs is $\boxed{\begin{array}{lcl} x_1 & = & 0 \\ x_2 & = & \frac{2}{5} \\ x_3 & = & \frac{9}{5} \end{array}}$ and at this point the optimal value of the objective function is attained and is $\frac{11}{5}$.

We will check our answer using LINDO 6.1.



```
LP OPTIMUM FOUND AT STEP      1

       OBJECTIVE FUNCTION VALUE

       1)      2.200000

 VARIABLE        VALUE          REDUCED COST
       X1         0.000000          2.600000
       X2         0.400000          0.000000
       X3         1.800000          0.000000


      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)         0.000000         -0.400000
       3)         0.000000         -0.200000

NO. ITERATIONS=       1

LP OPTIMUM FOUND AT STEP      0

       OBJECTIVE FUNCTION VALUE

       1)      2.200000

 VARIABLE        VALUE          REDUCED COST
       X1         0.000000          2.600000
       X2         0.400000          0.000000
       X3         1.800000          0.000000


      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)         0.000000         -0.400000
       3)         0.000000         -0.200000

NO. ITERATIONS=       0


       OBJECTIVE FUNCTION VALUE

       1)      2.200000

 VARIABLE        VALUE          REDUCED COST
       X1         0.000000          2.600000
       X2         0.400000          0.000000
       X3         1.800000          0.000000


      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)         0.000000         -0.400000
       3)         0.000000         -0.200000

NO. ITERATIONS=       0
```

Example 4.21: (M-technique)

$$\begin{array}{lllllllll}
\min & z & = & 4x_1 & + & x_2 & & & \\
\text{st} & & & & & & & & \\
3x_1 & + & x_2 & & & & = & 3 & (1) \\
4x_1 & + & 3x_2 & - & x_3 & & = & 6 & (2) \\
x_1 & + & 2x_2 & & + & x_4 & = & 4 & (3) \\
x_i & \geq & 0, & i & = & & 1,\ldots,4 & &
\end{array}$$

Note that we cannot find an obvious ibfs.

Add artificial "slacks" to (1) and (2) and appropriate penalties to the objective function.

$$
\begin{array}{llllllllll}
\min & z = & 4x_1 & +x_2 & & & +MR_1 & +MR_2 & & \\
\text{st} & & & & & & & & & \\
3x_1 & + & x_2 & & & & +R_1 & & = 3 & (1) \\
4x_1 & + & 3x_2 & -x_3 & & & & +R_2 & = 6 & (2) \\
x_1 & + & 2x_2 & & +x_4 & & & & = 4 & (3)
\end{array}
$$

$$x_i \geq 0, \quad i = 1,\ldots,4 \qquad R_j \geq 0 \quad j = 1,2$$

We solve this by the simplex method.

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Sol'n.	Tableau
$-z$	4	1	0	M	M	0	0	
R_1	3	1	0	1	0	0	3	
R_2	4	3	-1	0	1	0	6	
x_4	1	2	0	0	0	1	4	
$-z$	$4-7M$	$1-4M$	M	0	0	0	$-9M$	"0"
R_1	3	1	0	1	0	0	3	$3/3=1$
R_2	4	3	-1	0	1	0	6	$6/4=1.5$
x_4	1	2	0	0	0	1	4	$4/1=4$
$-z$	0	$-1-5M$	M	$\frac{7M-4}{3}$	0	0	$-2M-4$	"1"
x_1	1	$1/3$	0	$1/3$	0	0	1	$1/\frac{1}{3}=3$
R_2	0	$5/3$	-1	$-4/3$	1	0	2	$2/\frac{5}{3}=6/5$
x_4	0	$5/3$	0	$-1/3$	0	1	3	$3/\frac{5}{3}=9/5$

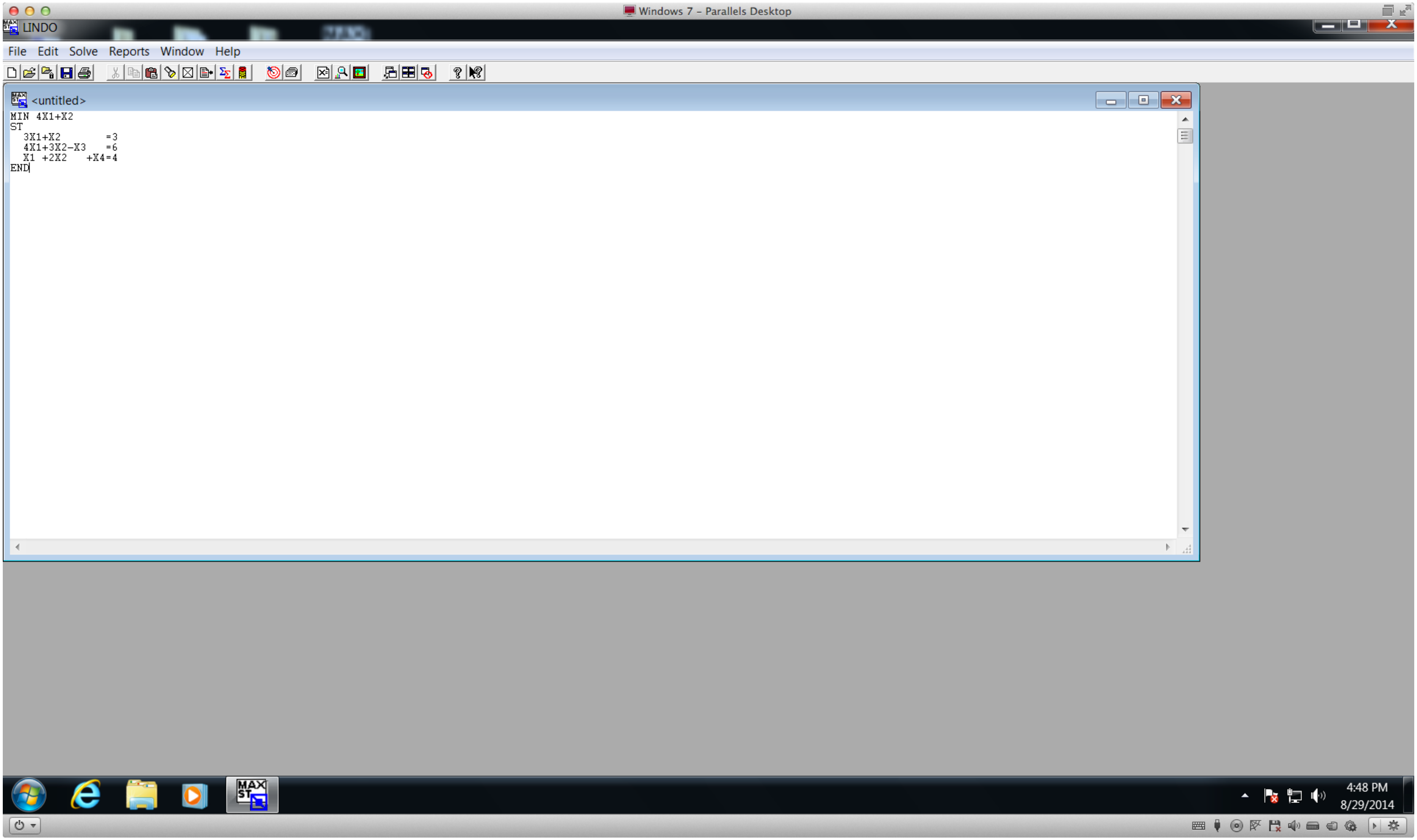
Basic	x_1	x_2	x_3	R_1	R_2	x_4	Sol'n	Tableau
$-z$	0	0	$-\frac{1}{5}$	$\frac{-8+5M}{5}$	$\frac{1+5M}{5}$	0	$-\frac{18}{5}$	"2"
x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{-1}{5}$	0	$\frac{3}{5}$	$\frac{3/5}{1/5}=3$
x_2	0	1	$\frac{-3}{5}$	$\frac{-4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$	
x_4	0	0	1	1	-1	1	1	$1/1=1$
$-z$	0	0	0	$\frac{-7+5M}{5}$	M	$\frac{1}{5}$	$-\frac{17}{5}$	"3"
x_1	1	0	0	$\frac{2}{5}$	0	$\frac{-1}{5}$	$\frac{2}{5}$	
x_2	0	1	0	$-\frac{1}{5}$	0	$\frac{3}{5}$	$\frac{9}{5}$	
x_3	0	0	1	1	-1	1	1	optimal

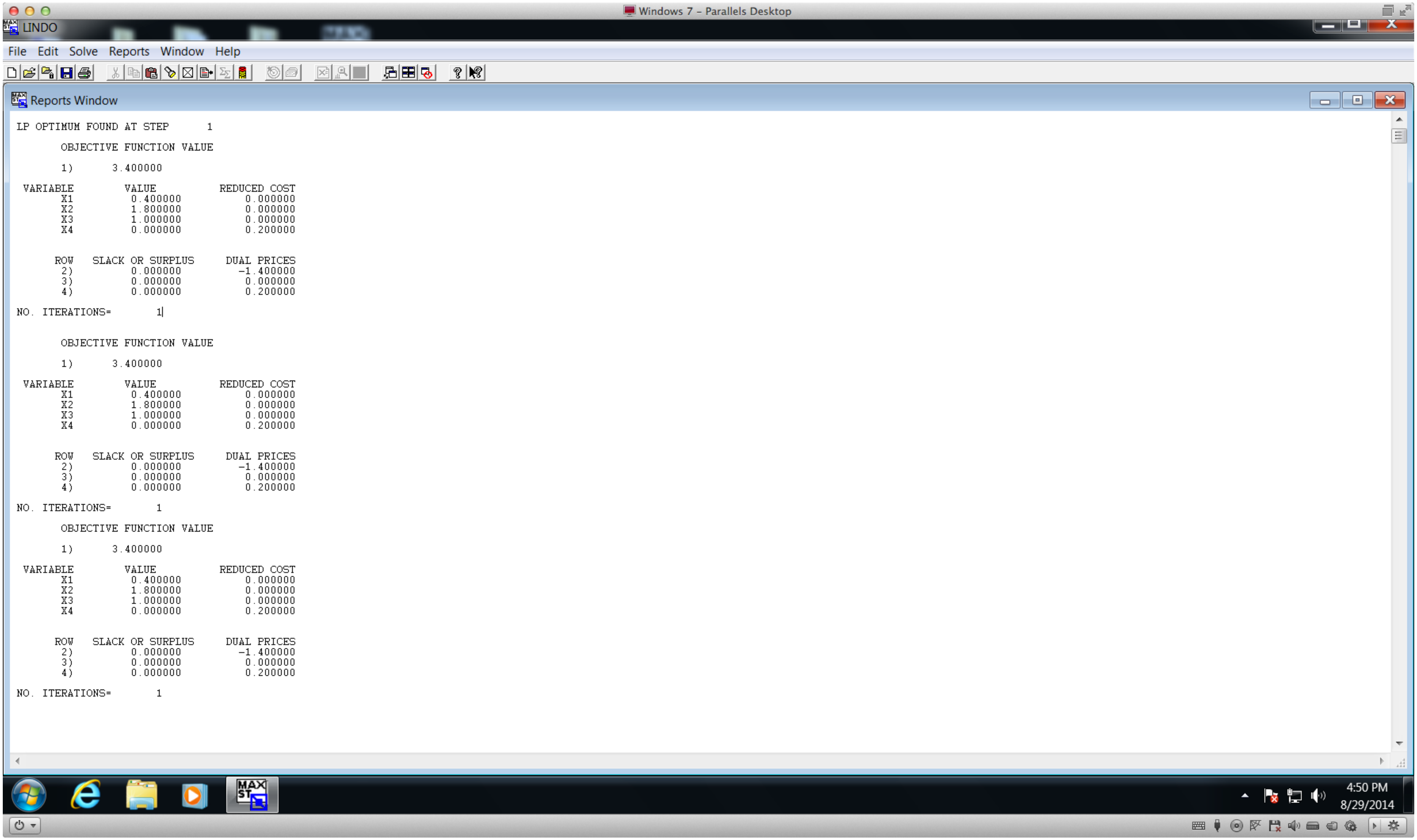
END

since this last tableau is optimal. The optimal bfs is

$$\boxed{\begin{aligned} x_1 &= \frac{2}{5} \\ x_2 &= \frac{9}{5} \\ x_3 &= 1 \\ x_4 &= 0 \end{aligned}}$$ with the optimal objective function (minimal) equaling $\frac{17}{5}$.

We can check our results with LINDO.


Windows 7 – Parallels Desktop
LINDO
File Edit Solve Reports Window Help
<untitled>
MIN 4X1+X2
ST
3X1+X2 =3
4X1+3X2-X3 =6
X1 +2X2 +X4=4
END
4:48 PM
8/29/2014


Windows 7 - Parallels Desktop
LINDO
File Edit Solve Reports Window Help
Reports Window
LP OPTIMUM FOUND AT STEP 1
OBJECTIVE FUNCTION VALUE
1) 3.400000
VARIABLE VALUE REDUCED COST
X1 0.400000 0.000000
X2 1.800000 0.000000
X3 1.000000 0.000000
X4 0.000000 0.200000
ROW SLACK OR SURPLUS DUAL PRICES
2) 0.000000 -1.400000
3) 0.000000 0.000000
4) 0.000000 0.200000
NO. ITERATIONS= 1
OBJECTIVE FUNCTION VALUE
1) 3.400000
VARIABLE VALUE REDUCED COST
X1 0.400000 0.000000
X2 1.800000 0.000000
X3 1.000000 0.000000
X4 0.000000 0.200000
ROW SLACK OR SURPLUS DUAL PRICES
2) 0.000000 -1.400000
3) 0.000000 0.000000
4) 0.000000 0.200000
NO. ITERATIONS= 1
OBJECTIVE FUNCTION VALUE
1) 3.400000
VARIABLE VALUE REDUCED COST
X1 0.400000 0.000000
X2 1.800000 0.000000
X3 1.000000 0.000000
X4 0.000000 0.200000
ROW SLACK OR SURPLUS DUAL PRICES
2) 0.000000 -1.400000
3) 0.000000 0.000000
4) 0.000000 0.200000
NO. ITERATIONS= 1
4:50 PM
8/29/2014

G. Degeneracy

Def. 4.21: a basic feasible solution (bfs) is called **DEGENERATE** iff at least one of the basic variables is 0.

Why do we concern ourselves with degenerate bfs?

Answer: In the Simplex Algorithm, at each stage we would like to strictly *improve* the value of the objective function. However, in the case of degenerate bfs, no improvement occurs. (The objective value remains the same.) Thus when degenerate bfs exist, we have the theoretical problem of proving that the algorithm converges (i.e. that it stops after finitely many steps. In fact, it might not! We might encounter **CYCLING** as in the following example.

Example 4.22(cycling):

$$\min \quad z = -\frac{3}{4}x_1 + 150x_2 - \frac{1}{50}x_3 + 6x_4$$

st

$$\begin{aligned}
&\frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 + x_5 = 0 \quad (1)\\
&\frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 + x_6 = 0 \quad (2)\\
&x_3 + x_7 = 1 \quad (3)\\
&x_i \geq 0, \quad i = 1,\ldots,7
\end{aligned}$$

We look at the reduced cost row and the basic variables having value 0.

Tableau	cur x_1	$-z$ x_2	row x_3	x_4	x_5	x_6	x_7	Basic Variables having value 0	Sol'n
0	$-\frac{3}{4}$	150	$-\frac{1}{50}$	6	0	0	0	x_5, x_6	0
1	0	-30	$-\frac{7}{50}$	33	3	0	0	x_1, x_6	0
2	0	0	$-\frac{2}{25}$	18	1	1	0	x_1, x_2	0
3	$\frac{1}{4}$	0	0	-3	-2	-3	0	x_3, x_2	0
4	$-\frac{1}{2}$	120	0	0	-1	1	0	x_3, x_4	0
5	$-\frac{7}{4}$	330	$\frac{1}{50}$	0	0	-2	0	x_5, x_4	0
6	$-\frac{3}{4}$	150	$-\frac{1}{50}$	6	0	0	0	x_5, x_6	0

Note that row 6 and row 0 are the same so we have cycled back to our initial bfs.

Why does this situation arise in the simplex tableau?

Answer: Whenever there is a tie for which variable will **exit**, then there will be a degenerate bfs at the next iteration. This is because the exit criterion, which is imposed to maintain feasibility, will push both (or all) possible candidates for entering to 0 and so at least one basic variable is 0.

Note: Use LINDO to solve this problem. (It will!) Cycling can be overcome, as we will soon see.

Def. 4.22: A bfs in n-space is called OVER-CONSTRAINED iff more than n hyperplanes intersect at this point.

Example 4.23:

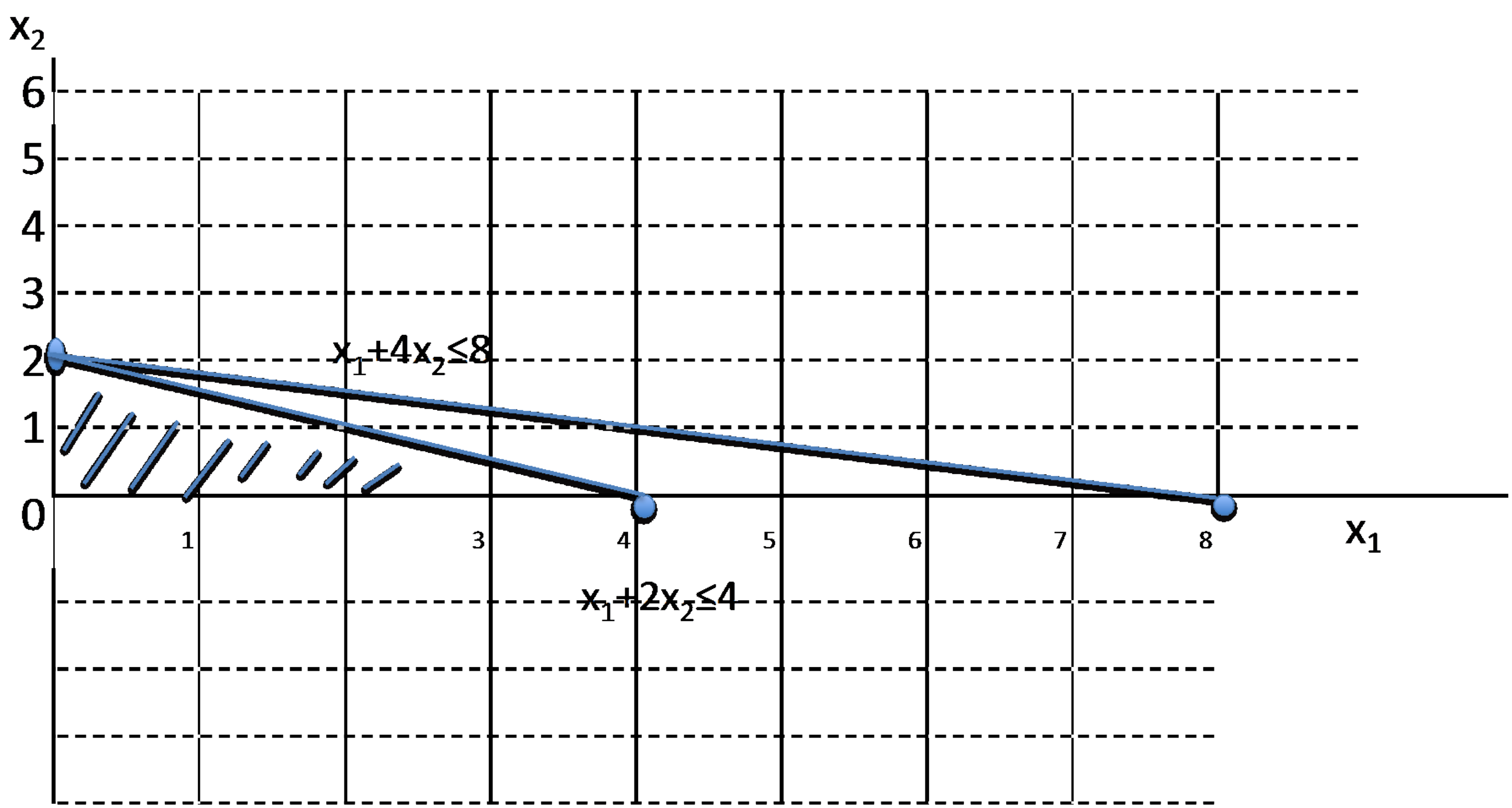
$$\max \quad z = 3x_1 + 9x_2$$

st

$$x_1 + 4x_2 \le 8 \quad (1)$$

$$x_1 + 2x_2 \le 4 \quad (2)$$

$$x_i \ge 0, \quad i = 1,2$$



x2
6
5
4
3
2
1
0
1
3
4
5
6
7
8
x1
x1+4x2≤8
x1+2x2≤4

Basic	x_1	x_2	s_1	s_2	Sol'n.	Tableau	
$-z$	-3	-9	0	0	0	"0"	
s_1	1	4	1	0	8	$8/4=2$	tie
s_2	1	2	0	1	4	$4/2=2$	tie
$-z$	$-\frac{3}{4}$	0	$\frac{9}{4}$	0	18	"1"	
x_2	$\frac{1}{4}$	1	$\frac{1}{4}$	0	2	$2/\frac{1}{4}=8$	
s_2	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	$0/\frac{1}{2}=0$	x_1 enters
$-z$	0	0	$\frac{3}{2}$	$\frac{3}{2}$	18	"2"	
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	2		optimal
x_1	1	0	-1	2	0		

Note that iterations "1" and "2" give the same solution (0,2,0,0) and z= 18. The only difference is tha in iteration "1" the basic variables are x_2, s_2 while in iteration "2" the basic

variables are x_2, x_1. Note that at optimality x_1 is a degenerate variable and we have a degenerate bfs.

Question 1: How can we spot a potential degeneracy?
Rule: If the coefficient of the entering variable of an "=0" equation is strictly positive, the next bfs will be degenerate with no change in the value of the objective function.

Question 2: Can we stop when we first spot a degeneracy?
Answer: No. Sometimes we only encounter a temporary degeneracy.

Question 3: How should we handle degenerate bfs in general?
Answer: LP procedures tend to ignore degeneracies since cycling rarely occurs. (Example 4.22 was a didactic example – that is, for the purpose of teaching.)

From a theoretical standpoint, degeneracies may be eliminated by one of three methods

(1) random choice of entering variable in case of tie

(2) lexicographic ordering

(3) perturbation method

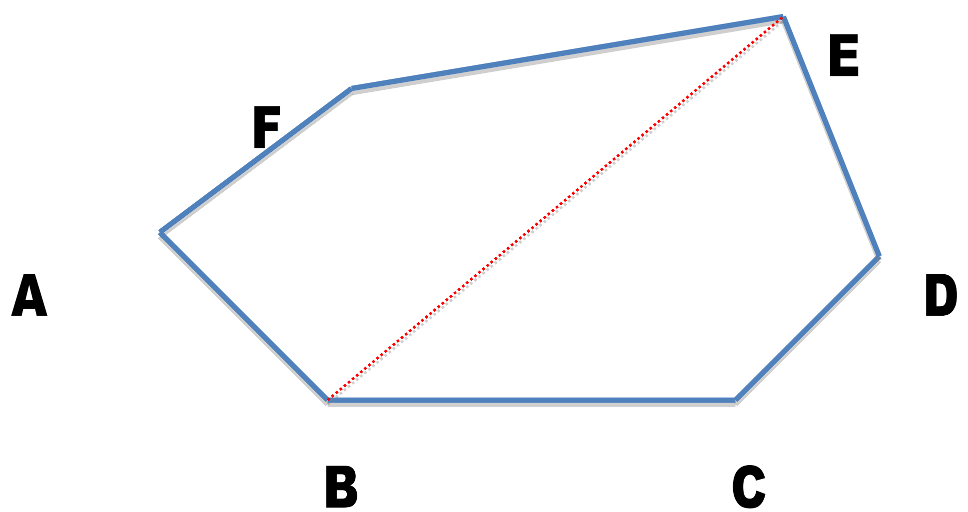
H. Non-Uniqueness of Optimal Solution

Question: **How can we tell if there is more than one optimal solution?**

Answer: **We note the following theorem which tells us that when there exist multiple optimal solutions then there must be adjacent optimal bfs.**

Theorem 4.3: **If an LP problem has more than one optimal solution, then it must have adjacent optimal bfs.**

Proof: **Consider the following polyhedron:**



Suppose B and E are obfs for the LP. Then the value of the linear objective function along the line BE is constant at the optimal value and this implies that the objective function is constant (at the optimal value) on the entire polyhedron ABCDEF. So, in particular, A, B and C are adjacent obfs.

What we have shown so far is that if an LP has optimal non-adjacent bfs on any face, then the entire face is optimal and so the LP has adjacent obfs.

If the LP has two obfs which are not on the same face, then joining them as we did above, gives a line on which the objective function (linear) is constant and passes through the interior of the constraint region. Hence the LP has an optimal interior solution which implies that all points in the constraint set are optimal.

The only other situation is that the two given obfs are adjacent.

Question: Assuming non-degeneracy, how does this show up in the Simplex Algorithm?
Rule: If at the optimal tableau, the reduced cost coefficient of a non-basic variable is 0, then the optimal solution is not unique. (This is because the value of the non-basic variable may be increased from 0 to a positive value without changing the value of the objective function (though the value of the other variables will be changed.)
Note that this method gives an adjacent obfs.

Note, in general, that if only one variable is leaving the basis and another variable is entering the basis, these two bfs are adjacent (or the same).

Example 4.24:

$$\text{Max} \quad z = 2x_1 + 4x_2$$

st

$$x_1 + 2x_2 \leq 5 \quad (1)$$

$$x_1 + x_2 \leq 4 \quad (2)$$

$$x_i \geq 0 \quad i = 1,2$$

which in standard form becomes

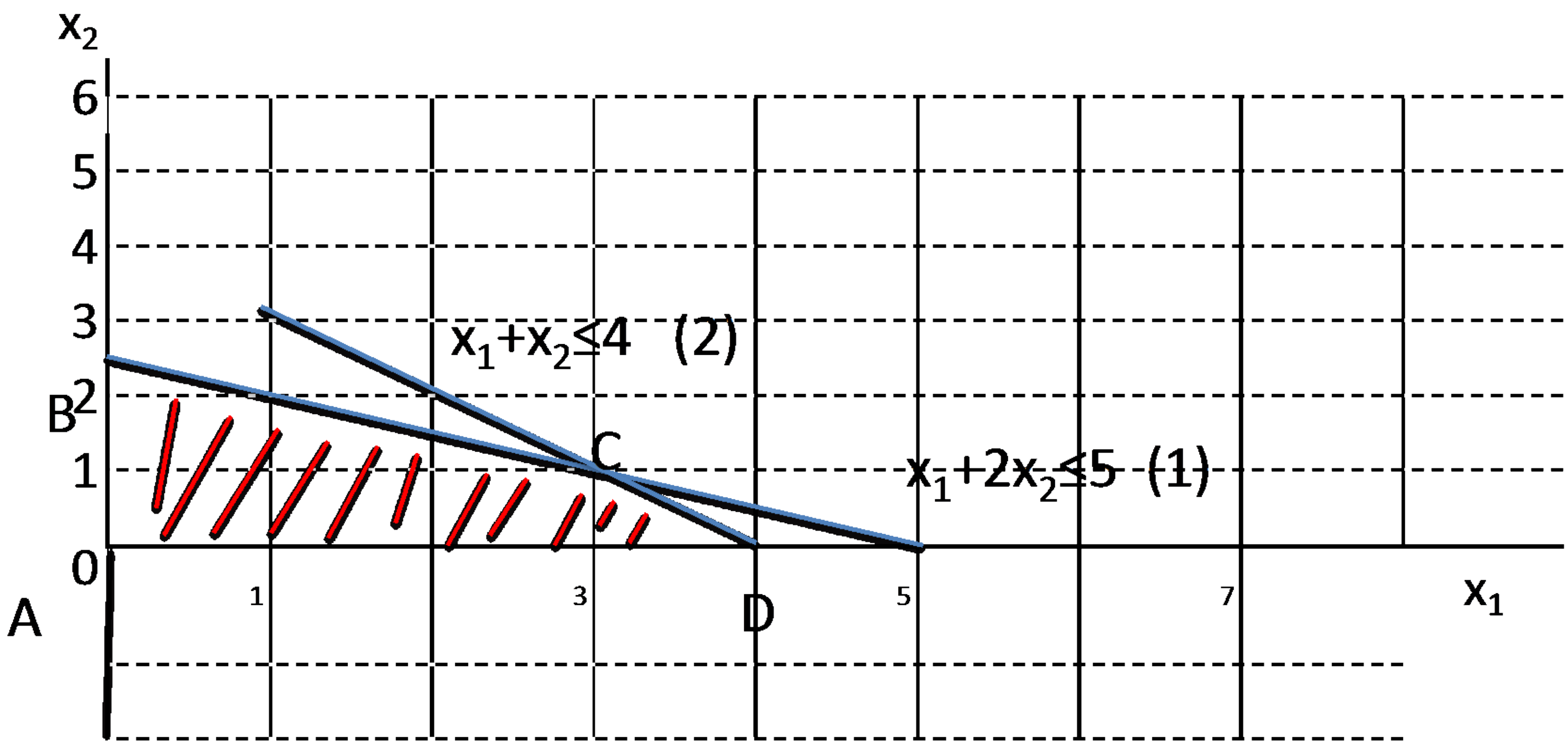
$$\text{Min} \quad z = -2x_1 - 4x_2$$

st

$$x_1 + 2x_2 + s_1 = 5 \quad (1)$$

$$x_1 + x_2 + s_2 = 4 \quad (2)$$

$$x_i \geq 0 \quad i = 1,2 \quad s_j \geq 0 \quad j = 1,2$$



x_2
6
5
4
3
2
1
0
B
A
C
D
$x_1+x_2\leq 4$ (2)
$x_1+2x_2\leq 5$ (1)
1
3
5
7
x_1

We will see that B and C are both obfs.

Basic	x_1	x_2	s_1	s_2	Sol'n.	Tableau
$-z$	-2	-4	0	0	0	"0"
s_1	1	2	1	0	5	$\frac{5}{2}$
s_2	1	1	0	1	4	$\frac{4}{1}$
$-z$	0	0	2	0	10	"1" optimal
x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{2}$	$\frac{\frac{5}{2}}{\frac{1}{2}} = 5$
s_2	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{\frac{3}{2}}{\frac{1}{2}} = 3$
$-z$	0	0	2	0	10	"2" adjacent optimal
x_2	0	1	1	-1	1	
x_1	1	0	-1	2	3	

I. Unbounded Solutions and Unbounded Solution Space

Rule 1: If at any stage, the column of coefficients under a variable (other than the cost coefficient) are negative or 0, then the solution space is unbounded in the direction of that variable.

Rule 2: If under Rule 1, the reduced cost is strictly less than 0, then there is no optimal solution; i.e. the LP is unbounded below.

END OF SECTION 4